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Similarity rules for thin aerofoils in non-stationary subsonic flows

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Similarity rules are constructed for the load distributions induced on a thin two-dimensional wing at subsonic speeds by sinusoidal gusts whose wave fronts are at an angle to the leading edge of the wing. It is shown that these rules divide into two groups according to the value of a parameter dependent on the Mach number and the angle between the gust front and the wing. The similarity rules for each group relate all the members of the group to a simpler problem whose solution can be found by existing methods. The similarity between the two groups is also discussed in terms of the two methods of solution available and it is shown that each method of solution is applicable in all cases.

1. Introduction

The calculation of the load distributions induced on a wing in a turbulent airstream usually proceeds from the simpler loading problem associated with the passage of a similar wing through a sinusoidal gust whose wave fronts are at an arbitrary angle to the leading edge of the wing. Under the conditions, outlined below, for which linear analysis is realistic the loading induced by an oblique sinusoidal gust is a double Fourier transform (with respect to time and one spatial co-ordinate) of the loading induced by turbulence or other more general gusts. As a result this problem, referred to hereafter as the oblique sinusoidal-gust problem, is relevant to a wide range of problems involving non-stationary aerodynamic loading. It is particularly relevant to calculations of the response of large aspect-ratio wings and helicopter rotor blades to turbulence and wakes, and in connexion with noise generation in compressors.

In general a gust encountered by an aerofoil will have local velocities of arbitrary magnitude and direction. Thus, as in figure 1 (*a*), the aerofoil flying horizontally at speed U and incidence α will experience a relative velocity $\mathbf{U}_\infty$ having components $U+u$ in the line of flight, w vertically and v in the spanwise direction. Within the framework of linearized aerofoil theory, for which the thickness of the aerofoil, its incidence and camber and the gust velocities are all assumed small, the effect of the spanwise velocity component v may be neglected. Neglecting second-order small quantities the situation can be simplified to that shown in figure 1 (*b*), and two first-order components of loading identified. These are a stationary component due to the mean relative upwash arising from incidence and camber, and a non-stationary component due to the upwash w.

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Under the condition $|w| \ll |u|$, the only non-stationary component may be a second-order small quantity dependent on the combination $u\alpha$. This, however, divides into a quasi-stationary component resulting from the total chordwise velocity $U+u$, which is readily calculable from stationary aerofoil theory, and a non-stationary component due to the relative upwash $u\alpha$ which may be treated like w. When the problem is linear these load distributions may be calculated separately and summed to give the whole loading. In this paper we are concerned only with the non-stationary component arising from the upwash w normal to the plane of the aerofoil.

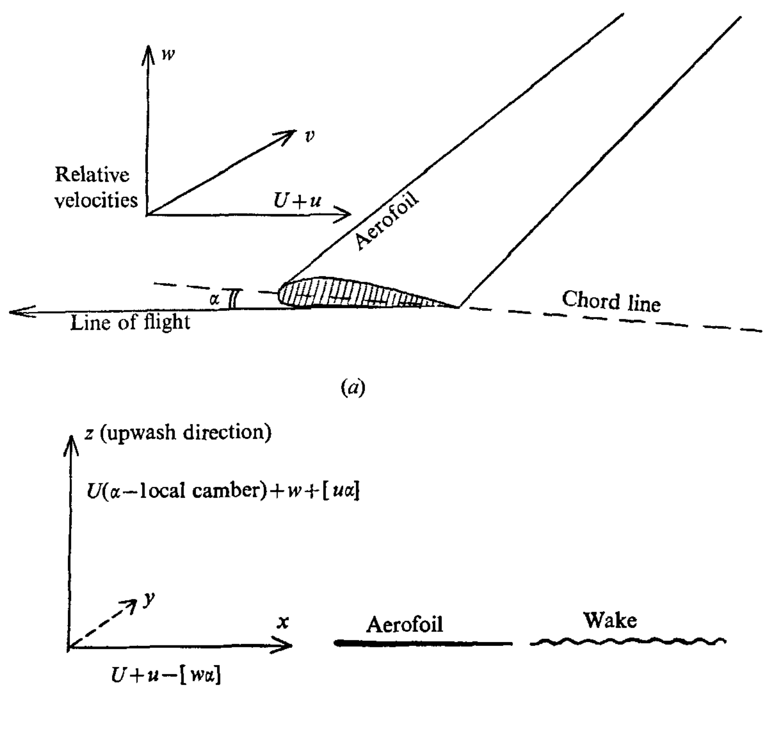

FIGURE 1. (*a*) Relative velocity components of a gust. (*b*) Linearized velocity components.

The oblique-sinusoidal-gust problem may be visualized as a corrugated sheet of vertical velocities travelling at speed U in the plane of the aerofoil as shown in figure 2. The corrugations have wavelengths $2\pi/\lambda$ and $2\pi/\mu$ in the chordwise and spanwise directions respectively and lie at an angle $(\tan^{-1}\mu/\lambda)$ to the leading edge of the aerofoil. Primarily we are concerned here with the unswept subsonic wing, in which case the velocity of translation U, assumed uniform and subsonic, is at right angles to the leading edge of the aerofoil.

The purpose of the present paper is to simplify this generalized problem, outwardly dependent on three non-dimensional parameters (Mach number and two wave-number parameters) by relating it to one or other of two lower-order problems involving only two out of the three parameters. These simpler problems are special cases of the general problem, being respectively the case when the flow is incompressible and the gust oblique, and the compressible case for which the gust wave fronts are parallel to the leading edge of the wing. In the notation of figure 2 these are respectively the cases ($M = 0, \lambda, \mu$) and ($M, \lambda, \mu = 0$) and are referred to hereafter as incompressible oblique and compressible two-dimensional.

The simplest convected gust problem of all, that for which the flow is both incompressible and two-dimensional, has an analytical solution (Sears 1941). This

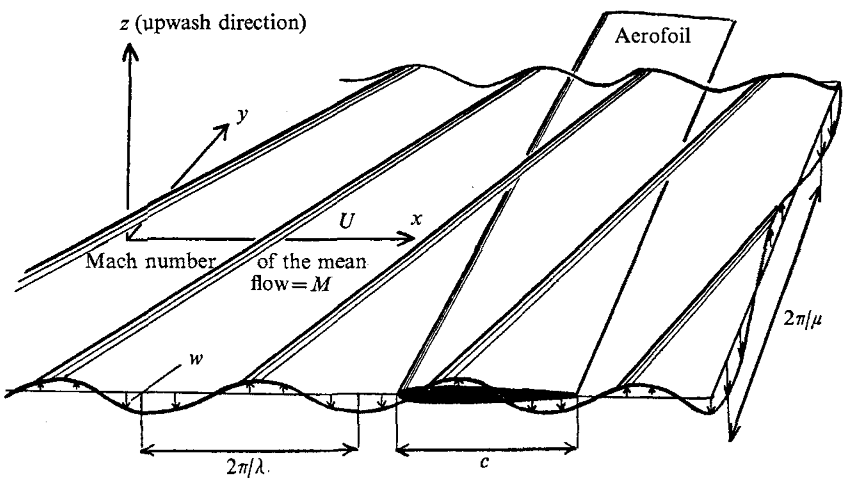

FIGURE 2. Diagrammatic drawing of an oblique sinusoidal gust.

solution depends upon the linearization assumptions of thin-aerofoil theory with additional assumptions that both the gust and the wake vorticity are convected as frozen patterns at the free-stream speed and that rotationality in the free stream is unimportant. These assumptions are carried over into the three-dimensional compressible-flow analysis given here and therefore limit the range of validity of the analysis to small disturbance flows over thin aerofoils.

In the notation of figure 2, the conditions for linearization of the time-dependent equations for the velocity potential in subsonic flow are set out by Miles (1959, p. 8, table 1). They are (a) $\delta^{\frac{2}{3}} \ll 1 - M$, where δ is the ratio of a characteristic displacement of the streamlines to the chord c of the aerofoil, (b) $k = O(\beta^2/M^2)$, and (c) $\nu = O(\beta)$. Here $\beta^2 = 1 - M^2$, and k and ν are the reduced frequency parameters $\frac{1}{2}\lambda c$ and $\frac{1}{2}\mu c$ respectively.

Corrections to the potential-flow analysis for free-stream vorticity are discussed in a paper by Hunt (1969). This paper implies that for thin-aerofoil problems for which $\delta \ll 1$, the free-stream vorticity may be neglected provided that variations in the mean-pressure field of the body are small compared with the total pressure in the free stream. This therefore limits the present analysis to lightly-loaded aerofoils.

The majority of aerofoil gust interactions and allied problems are concerned with the convection of either discrete gusts or random disturbances, i.e. turbulence, past the aerofoil. It is usual in dealing with such phenomena to assume that these disturbances remain relatively frozen in a frame of reference moving at the mean velocity of the fluid. That is, changes in the convected upwash pattern in the plane of the aerofoil and the wake only occur over distances much greater than the chord of the aerofoil. This is referred to as the frozen convection assumption.

This assumption is valid for turbulent flows when the times for significant viscous dissipation and turbulent diffusion of an eddy are much larger than the time for the eddy to travel one chord length. Following Hunt's (1969) analysis this yields for the thin aerofoil gust problem the conditions that

$$[\overline{w^2}/U^2]^{\frac{1}{2}} \ll 1/k \quad \text{or} \quad k\delta \ll 1$$

and

$$[\overline{w^2}/U^2]^{\frac{1}{2}} \ll 1 \quad \text{or} \quad \delta \ll 1.$$

The second condition is already a requirement for linearization.

It is not impossible to analyze the effect of sinusoidal gusts convected at speeds different from that of the free stream, although convection at the free-stream velocity will be assumed throughout the analysis given below. But the validity of linearization, and hence its associated conditions, are vital to the analysis of aerofoils in turbulent flows, since without this particular assumption Fourier analysis cannot be used and the problem passes beyond the limits of present-day tractability.

For the incompressible two-dimensional problem Sears adopted as his model for the potential flow, a distribution of vortex singularities over the aerofoil and in the wake. The singularities in the wake are required to conserve vorticity in the entire field. In this way a solution can be obtained by satisfying the upwash boundary condition on the chord line of the aerofoil in terms of these singularities. It is important to distinguish these vortex singularities on the aerofoil and in its wake from the free-stream vorticity which is neglected in this analysis. The same approach was used by the present author (Graham 1970*a*) to obtain the solution for the incompressible oblique gust (the two-wave-number problem). That paper was mainly concerned with an efficient reduction method of solving the singular integral equation for the upwash boundary condition and computing a numerical series solution. An approximate analytical solution to the same problem, accurate asymptotically at high and low wave-numbers, has also been obtained by Filotas (1969).

The unsteady, compressible, two-dimensional thin-aerofoil problem, as a flutter problem, was first analyzed by Possio (1938). He formulated it in terms of an integral equation for the load distribution on the aerofoil and gave an approximate solution. This approach led to several successively refined approximate methods of solving the problem, of which one of the best is that of Fettis (1952). His method isolates the singular part of the integral equation, approximates the remainder of the kernel function linearly and exactly inverts the resulting integral equation. This process constitutes in effect a first approximation to the complete numerical method of solution developed for the incompressible oblique gust case.

Reissner (1951) gave an analytic solution of the same compressible two-dimensional problem, again for a flutter rather than gust boundary condition. By transforming the linearized perturbation potential equation into elliptic co-ordinates he found a separable circulatory solution satisfying the boundary conditions. This method is exact but very lengthy to compute. However, the transformation used in this method forms the basis of the similarity rules which are presented below. These similarity rules relate the general three-parameter problem (compressible oblique) to the two simpler two-parameter problems, incompressible oblique and compressible two-dimensional.

In the second part of the analysis given below, the similarity between these latter two problems is discussed and it is shown that the same methods of solution are applicable to both.

2. Similarity rules for thin aerofoils in subsonically convected oblique sinusoidal upwash patterns

By making the assumptions listed in § 1 a perturbation velocity potential ϕ may be defined in the usual way. That is, if $\mathbf{U}_\infty$ is the undisturbed free-stream velocity, the velocity field in the vicinity of the aerofoil is $\mathbf{U}_\infty + \nabla\phi$, where $|\nabla\phi| \ll |\mathbf{U}_\infty|$ except near discontinuities of the aerofoil's surface slope. On substituting ϕ into the continuity equation and assuming that the linearization process is valid, the equation

$$(1-M^2)\,\phi_{xx}+\phi_{yy}+\phi_{zz}-(2M/a)\,\phi_{xt}-(1/a^2)\,\phi_{tt} = 0 \tag{1}$$

is obtained. In this equation suffices denote differentiation with respect to the appropriate co-ordinates and a is the speed of sound in the free stream.

For a thin flat aerofoil lying in the plane $z = 0$ and with its mid-chord line along the y axis, the linearized boundary condition that the velocity is tangential to the aerofoil surface is

$$(\mathbf{U}_\infty . \mathbf{n}+\phi_z) = 0, \quad z = 0, \quad -\tfrac{1}{2}c \leqslant x \leqslant \tfrac{1}{2}c; \tag{2}$$

$\mathbf{n}$ is here a unit vector parallel to the z axis.

For an oblique sinusoidal vertical-gust convected at the mean free-stream velocity U

$$\mathbf{U}_\infty = \{U, 0, w_0 \exp[i(\omega t-\lambda x-\mu y)]\}, \quad \text{where} \quad \omega = \lambda U.$$

In this case equation (2) becomes

$$\phi_z = -w_0 \exp[i(\omega t-\lambda x-\mu y)], \quad z = 0, \quad -\tfrac{1}{2}c \leqslant x \leqslant \tfrac{1}{2}c. \tag{3}$$

Equations (1) and (3) are compatible with a potential ϕ of the form

$$\phi = \phi'(x,z) \exp[i(\omega t-\mu y)].$$

Substituting this into (1) gives

$$(1-M^2)\,\phi'_{xx}+\phi'_{zz}-(2i\omega M/a)\,\phi'_x+((\omega^2/a^2)-\mu^2)\,\phi' = 0. \tag{4}$$

Now, following Reissner (1951), we make the transformations

$$\Phi = \exp[-ikM^2x^*/\beta^2]\,\phi', \quad x^* = 2x/c, \quad z^* = 2\beta z/c \quad \text{and} \quad y^* = 2y/c.$$

Writing $\nu = \frac{1}{2}\mu c$, $k = \frac{1}{2}\omega c/U = \frac{1}{2}\lambda c$ and $K = kM/\beta^2$ we obtain the equation for the potential in the transformed 'Prandtl–Glauert' plane

$$\Phi_{x^*x^*} + \Phi_{z^*z^*} + K^2(1-1/\theta^2)\,\Phi = 0, \tag{5}$$

where θ is the gust parameter $kM/\nu\beta$.

The boundary condition (3) on the aerofoil is, in this plane,

$$\Phi_{z^*} = -(\tfrac{1}{2}w_0 c/\beta)\exp[-ikx^*/\beta^2] \quad (z^* = 0, -1 \leqslant x^* \leqslant 1). \tag{6}$$

With the assumptions made earlier the equation for the pressure may also be linearized. The pressure coefficient C_p derived from this linearized pressure equation is given in the transformed system by

$$C_p = -(4/Uc)\exp[ikM^2x^*/\beta^2]\{ik\Phi/\beta^2 + \Phi_{x^*}\}. \tag{7}$$

The flow model used in the above analysis represents the effect of the aerofoil as a potential perturbation of the free stream. This potential is discontinuous across the aerofoil and, since the flow is non-stationary, also across the wake, assumed here to lie downstream of the aerofoil in the $z = 0$ plane. Boundary conditions to define the problem are therefore not only necessary over the aerofoil but also over its wake and at infinity. The first of these boundary conditions has already been given. The second is that Δp, the pressure discontinuity across $z = 0$, is zero over the wake. Or, from (7),

$$\Delta(ik\Phi/\beta^2 + \Phi_{x^*}) = 0 \quad (x^* > 1). \tag{8}$$

The third boundary condition, at infinity, takes two different forms depending on whether the potential equation (5) is elliptic or hyperbolic. In the former case ($\theta \leqslant 1$) the boundary condition may be taken in the form

$$\Phi \to 0 \quad \text{as} \quad (x^{*2} + z^{*2}) \to \infty. \tag{9a}$$

That is disturbances die away at infinity. But in the latter case ($\theta > 1$) it becomes necessary to prescribe that signals are radiated outwards from the aerofoil toward infinity without reflexion. In this case the boundary condition can be taken in the form

$$\exp[iK(1-1/\theta^2)^{\frac{1}{2}}r] \quad \Phi \to 0 \quad \text{as} \quad z^* \to \pm\infty, \tag{9b}$$

where $r^2 = x^{*2} + z^{*2}$. Finally, for any subsonic flow the aerofoil circulation can only be fixed by specifying the additional Kutta condition that $\Delta p = 0$ at the trailing edge. Therefore (8) may be rewritten

$$\Delta(ik\Phi/\beta^2 + \Phi_{x^*}) = 0, \quad x^* \geqslant 1. \tag{10}$$

We now consider the different types of solutions of (5) together with these boundary conditions, corresponding to values of θ greater than, less than or equal to one.

2.1. *Subcritical flows: $\theta \leqslant 1$*

Equation (5) has the form $\Phi_{x^*x^*} + \Phi_{z^*z^*} - \chi^2\Phi = 0$, ($\chi$ real) for all flow fields in this group. It follows from this that given any member of this group it is possible to find an infinite number of other members having similar Φ distributions, differing at most by a constant multiple. The requirement for this to be so is that they all give rise to the same potential equation and boundary conditions for Φ. It is clear therefore from the analysis above that these similar flows must lie in (M, k, ν) space on the lines χ = constant, k/β^2 = constant. Each member of the group will have only one such line passing through it and each line will intersect the surface $\theta = 0$ once only. This surface $\theta = 0$ consists of the incompressible oblique members of the group. Therefore each member of the group is similar to an incompressible oblique problem ($\theta = 0$, subscripted 0 below), i.e. $M_0 = 0$ and w_0 (upwash) $= \exp[i(\omega_0 t - \lambda_0 x - \mu_0 y)]$. The set of similarity rules relating the flows to satisfy equations (5), (6) and (10) are

$$\left.\begin{aligned} M_0 &= 0, \\ k_0 &= k/\beta^2, \\ \nu_0 &= \nu(1-\theta^2)^{\frac{1}{2}}/\beta, \end{aligned}\right\} \tag{11}$$

and therefore from (7) the relationship between the loading coefficients per unit upwash is

$$\begin{aligned} C_{\Delta p}(M,k,\nu) = C_{\Delta p_0}&(0, k/\beta^2, \nu(1-\theta^2)^{\frac{1}{2}}/\beta) \\ &\times \frac{1}{\beta}\exp[i[kM^2x^*/\beta^2 + \nu\{(1-\theta^2)^{\frac{1}{2}}/\beta - 1\}y^*]]. \end{aligned} \tag{12}$$

The boundary condition (9) at infinity is automatically satisfied both for these cases and the supercritical cases which follow.

2.2. *Supercritical flows: $\theta \geqslant 1$*

Equation (5) has the form $\Phi_{x^*x^*} + \Phi_{z^*z^*} + \chi^2\Phi = 0$, and therefore in the case of the supercritical group each member of the group is similar to a compressible two-dimensional problem ($\theta = \infty$, subscripted ∞ below), i.e.

$$M_\infty \neq 0 \quad \text{and} \quad w_\infty = \exp[i(\omega_\infty t - \lambda_\infty x)].$$

The similarity rules for this group are

$$\left.\begin{aligned} M_\infty &= M(1-1/\theta^2)^{\frac{1}{2}}, \\ k_\infty &= k(1+\nu^2/k^2), \\ \nu_\infty &= 0, \end{aligned}\right\} \tag{13}$$

and

$$\begin{aligned} C_{\Delta p}(M,k,\nu) = C_{\Delta p_\infty}&\{M(1-1/\theta^2)^{\frac{1}{2}}, k(1+\nu^2/k^2), 0\} \\ &\times (1+\nu^2/k^2)^{\frac{1}{2}}\exp[i\nu(\nu x^*/k - y^*)]. \end{aligned} \tag{14}$$

Both sets of similarity rules overlap at $\theta = 1$ to give

$$\left.\begin{aligned} M_1 &= 0, \\ k_1 &= k/\beta^2, \\ \nu_1 &= 0, \end{aligned}\right\} \tag{15}$$

and

$$C_{\Delta p}(M,k,\nu) = C_{\Delta p_1}(0, k/\beta^2, 0)(1/\beta)\exp[i(kM^2x^*/\beta^2 - \nu y^*)], \tag{16}$$

i.e. at $\theta = 1$ the flows are similar to the incompressible two-dimensional case whose solution was given by Sears (1941).

2.3. *Remarks on the similarity rules*

For stationary flows (i.e. λ, $\omega = 0$) the appropriate similarity rule ($\theta < 1$) reduces to the Prandtl–Glauert transformation between corresponding compressible and incompressible flows. This transformation of the spanwise reduced frequency ν is correct to second order in θ for all flows in the subcritical group.

The physical significance of the parameter θ which characterizes a particular case as subcritical or supercritical may be seen as follows.

By a suitable transformation to moving axes it is possible to convert the non-stationary oblique-gust problem into a stationary problem (see figure 3). Filotas (1969) has approximately analyzed the incompressible oblique problem by this method. The flow is stationary relative to axes moving in the spanwise direction at the 'trace velocity' of the intersections of the nodal lines of the gust with the leading edge of the aerofoil. The trace velocity is Uk/ν relative to the

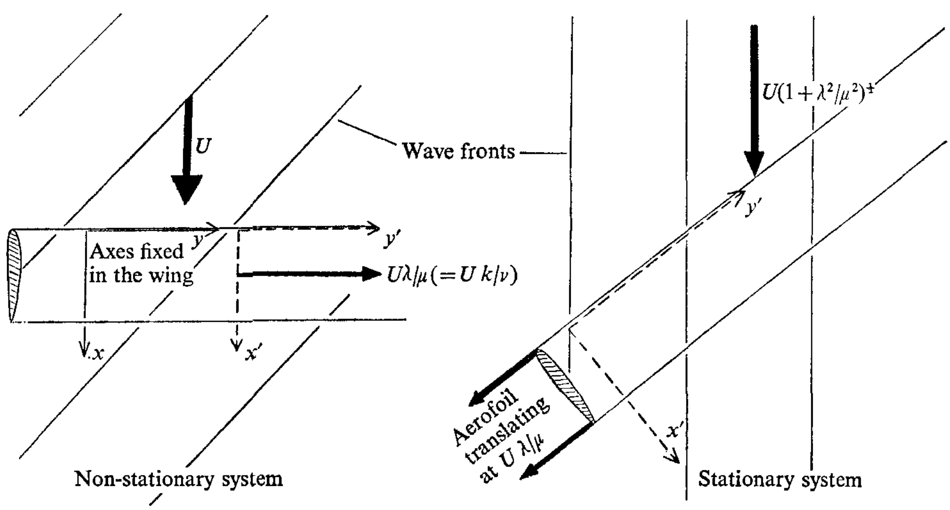

FIGURE 3. The transformation from a non-stationary to a stationary system.

aerofoil and therefore the velocity of the free stream relative to the moving axis system is $U(1+k^2/\nu^2)^{\frac{1}{2}}$. If, however, the fluid is compressible the free-stream Mach number relative to this system is $M' = M(1+k^2/\nu^2)^{\frac{1}{2}}$. Values of the gust parameter $\theta < 1$ or > 1 correspond respectively to values of this Mach number $M' < 1$ or > 1. Therefore under this transformation to moving axes the oblique gust problem is converted to the stationary problem of a twisted swept wing with subsonic edges in a subsonic or supersonic free stream. Alternatively, in the original co-ordinate system values of $\theta < 1$ or > 1 correspond respectively to flows in which the spanwise velocity of sound waves is greater or less than the spanwise trace velocity of the nodal lines of the gust.

Both the wake boundary condition (10) and the upwash boundary condition (6) define a (k, β) relation for similarity to be possible. In general it is only in those cases for which the 'free-stream-frozen-convection assumption' can be made that these two relations are the same (k/β^2 invariant). The reason that this is so in the frozen-convection problem is a result of the fact that both w over the aerofoil and ϕ over the wake are stationary in a reference frame moving with the free stream. This condition does not apply to the relative upwash for many classes of non-stationary flows, in particular those involving flutter. Similarity rules between flows having the same type of upwash cannot therefore be constructed in such cases.

The similarity rules above are for unswept aerofoils. But the same analysis can be applied to aerofoils of arbitrary sweep. In this case it is found that the same similarity rules apply if M and k are now defined respectively as the Mach number and reduced frequency in the direction normal to the leading edge and ν is the reduced frequency in the spanwise direction. Alternatively if M, k and ν are defined in terms of axes aligned with the free stream and Γ is the sweep angle

$$\theta = \left| \frac{(k\cos\Gamma - \nu\sin\Gamma)\, M\cos\Gamma}{(k\sin\Gamma + \nu\cos\Gamma)\,(1-M^2\cos^2\Gamma)^{\frac{1}{2}}} \right|.$$

The similarity rules are, in addition, not restricted to isolated aerofoils, but may also be applied to linear cascades with the appropriate additional similarity rule for the spacing ratios. Similarity rules may likewise be constructed for supersonic flight through gusts, but are not considered in this paper.

2.4. *The shape of the induced load distribution*

A further important consequence of the frozen-convection assumption implies a simplification in the expression for the load distribution.

Consider the function $\Pi(x^*, z^*) = (ik/\beta^2\Phi + \Phi_{x^*})$, which is equivalent to an acceleration potential in the Prandtl–Glauert plane. This function is continuous everywhere except with respect to z^* across the aerofoil and with respect to x^* at the leading edge. The equation for the potential Φ (5), together with its boundary condition equations (6), (9) and (10) may be rewritten in terms of Π as

$$\Pi_{x^*x^*} + \Pi_{z^*z^*} + K^2(1-1/\theta^2)\,\Pi = 0;$$

$$\Pi_{z^*} = 0, \begin{Bmatrix} z^* = 0_+ \\ z^* = 0_- \end{Bmatrix}, \quad -1 \leqslant x^* \leqslant 1;$$

$$\Pi = 0, \quad z^* = 0, \quad x^* \geqslant 1;$$

$$\left.\begin{aligned} &\Pi \to 0 \quad \text{as} \quad (x^{*2}+z^{*2}) \to \infty, \quad \theta \leqslant 1; \\ \text{or} \quad &\exp[iK(1-1/\theta^2)^{\frac{1}{2}} r]\,\Pi \to 0 \quad \text{as} \quad z^* \to \pm\infty, \quad r^2 = x^{*2}+z^{*2}, \quad \theta > 1; \end{aligned}\right\}$$

where 0_+, 0_- are understood to mean that a limit is taken as z^* approaches $z^* = 0$ from above and below respectively, in order to take into account the discontinuity across $z^* = 0$. Because these boundary conditions for Π are homogeneous there exist an infinite number of similar solutions to these equations. The reason for this is that the boundary condition for Π_{z^*} on the aerofoil is less strong than the original boundary condition (6) for Φ_{z^*} on the aerofoil. An additional boundary condition on Π is required to account for this. This condition can be taken in the form

$$\lim_{z^*\to 0} \int_{-\infty}^{-1} \exp[ik/\beta^2 x^*]\,\Pi_{z^*}\,dx^* = -\frac{w_0 c}{2\beta},$$

which fixes the remaining multiplicative constant in the solution for Π. In consequence $\Delta\Pi$ has the general form on the aerofoil

$$\Delta\Pi(x^*) = F(M,k,\nu)\, G\{x^*, K(1-1/\theta^2)^{\frac{1}{2}}\},$$

and therefore from (7)

$$C_{\Delta p}(x^*) = -(4/Uc)\, F(M,k,\nu) \exp[ikM^2x^*/\beta^2]\, G\{x^*, K(1-1/\theta^2)^{\frac{1}{2}}\}.$$

Applying this result to the similarity rules we obtain the following functional forms for the subcritical and supercritical load distributions:

$\theta \leqslant 1$:

$$C_{\Delta p}(M,k,\nu,x^*,y^*,t)$$
$$= (1/\beta) f_0\{0, k/\beta^2, (1-\theta^2)^{\frac{1}{2}}/\beta\}\, g_0\{\nu(1-\theta^2)^{\frac{1}{2}}/\beta, x^*\} \exp[i[\omega t + kM^2x^*/\beta^2 - \nu y^*]]$$

and $\theta \geqslant 1$:

$$C_{\Delta p}(M,k,\nu,x^*,y^*,t)$$
$$= (1+\nu^2/k^2)^{\frac{1}{2}} f_\infty\{M(1-1/\theta^2)^{\frac{1}{2}}, k(1+\nu^2/k^2), 0\}\, g_\infty\{(kM/\beta^2)(1-1/\theta^2)^{\frac{1}{2}}, x^*\}$$
$$\times \exp[i[\omega t + kM^2x^*/\beta^2 - \nu y^*]],$$

where all the independent variables of each function are listed in parentheses immediately after it. Since f_0 and f_∞ are independent of x^*, they may be taken as the appropriate lift coefficients $C_L(M',k',\nu')$. Their form is compatible with the additional boundary condition for Π given above.

For the case of an incompressible fluid this implies that the shape of the load distribution and hence the centre of lift depend only on the spanwise reduced frequency ν. Cicala (1951) showed that this was true for the incompressible two-dimensional problem ($M = 0, \lambda, \mu = 0$). From this there follows the surprising result that the load distribution induced on an aerofoil travelling through any frozen two-dimensional upwash pattern has at all times the flat-plate

$$[(1-x^*)/(1+x^*)]^{\frac{1}{2}}$$

or '$\cot\frac{1}{2}\theta$' shape, when compressibility is negligible.

2.5. *The sonic limit of the similarity rules*

The proper linearized equation for the unsteady potential at transonic speeds for $k = O(1)$ is

$$\phi_{yy} + \phi_{zz} - 2\phi_{xt} - \phi_{tt} = 0$$

(for example, Landahl 1961). An analytical result for the pressure distribution induced by an oblique sinusoidal gust in the limit $M = 1$, is readily derived from this equation as

$$C_{\Delta p}(M = 1, k, \nu, x^*, y^*, t)$$
$$= [2(1-i)\exp(ik)/\{\pi k(1+x^*)\}]^{\frac{1}{2}} \exp[i\{\omega t - \tfrac{1}{2}k(1-\nu^2/k^2)(1+x^*) - \nu y^*\}].$$

When $M \to 1$ the gust parameter $\theta \to \infty$ for all finite values of ν/k. Although equation (5) for the potential in the transformed plane is not strictly applicable in the transonic régime, the sonic limit of the supercritical similarity rule derived from this equation does give the correct relationships for the appropriate pressure distributions given above.

3. The similarity between the sub- and super-critical groups

A certain mathematical similarity also exists between the sub-critical and super-critical flows considered above. In view of the previous analysis it is sufficient to compare the two extreme cases, incompressible oblique and compressible two-dimensional.

For the first of these the equations governing Φ are

$$\left.\begin{aligned} &\Phi_{x^*x^*} + \Phi_{z^*z^*} - \nu^2\Phi = 0, \\ &\Phi_{z^*} = -\tfrac{1}{2}wc\exp[-ikx^*], \quad z^* = 0 \quad (-1 \leqslant x^* \leqslant 1), \\ &\Delta(ik\Phi + \Phi_{x^*}) = 0 \quad (x^* \geqslant 1), \\ &\Phi \to 0 \quad \text{as} \quad (x^{*2}+z^{*2}) \to \infty, \end{aligned}\right\} \tag{17}$$

and similarly for the second

$$\left.\begin{aligned} &\Phi_{x^*x^*} + \Phi_{z^*z^*} + (k^2M^2/\beta^4)\,\Phi = 0, \\ &\Phi_{z^*} = -(\tfrac{1}{2}wc/\beta)\exp[-ikx^*/\beta^2] \quad z^* = 0\,(-1 \leqslant x^* \leqslant 1), \\ &\Delta(ik/\beta^2\Phi + \Phi_{x^*}) = 0 \quad (x^* \geqslant 1), \\ &\exp[ikMr/\beta^2].\,\Phi \to 0 \quad r^2 = (x^{*2}+z^{*2}) \quad \text{as} \quad z \to \pm\infty. \end{aligned}\right\} \tag{18}$$

From these sets of equations it is at once evident that the two cases, subscripted 0 and ∞ as before, are identical for imaginary

$$\nu_0 = ik_\infty M_\infty/\beta_\infty^2 \tag{19}$$

and

$$k_0 = k_\infty/\beta_\infty^2, \tag{20}$$

provided the solutions have the required properties at infinity. In this case the relation between the loading coefficients per unit upwash is, from the linearized pressure equation (7)

$$C_{\Delta p_0} = \beta_\infty \exp[-ik_\infty M_\infty^2 x^*/\beta_\infty^2]\,C_{\Delta p}\exp[-i\nu_0 y^*]. \tag{21}$$

Alternatively by starting from Possio's formulation of the two-dimensional unsteady compressible flow integral equation and using the relationships (19) and (20) the integral equation for the incompressible oblique gust (Graham 1970*a*) is arrived at, as follows.

Watkins *et al.* (1955, equation B 18) have shown that Possio's integral equation can be written as

$$\frac{1}{4\pi}\int_{-1}^{1} \Delta p(x^*)\,\mathbf{K}(x_0^* - x^*)\,dx^* = w(x_0^*), \quad \text{the upwash.} \tag{22}$$

The kernel function $\mathbf{K}(x_0^* - x^*)$ is given by

$$\mathbf{K}(x) = -\frac{\pi k}{\beta}e^{-ikx}\left\{e^{ikx/\beta^2}\left[\frac{iM\,|x|}{x}H_1^{(2)}\left\{\frac{kM\,|x|}{\beta^2}\right\} - H_0^{(2)}\left\{\frac{kM\,|x|}{\beta^2}\right\}\right]\right.$$
$$\left. + \frac{2i}{\pi}\beta\log\left(\frac{1+\beta}{M}\right) + ik\int_0^x e^{ikx/\beta^2}H_0^{(2)}\left\{\frac{kM\,|x|}{\beta^2}\right\}dx\right\}, \tag{23}$$

where $H_n^{(2)}$ are Hankel functions. These Hankel functions can be replaced by the modified Bessel functions K_n by means of the identity

$$(2i/\pi)\, e^{\frac{1}{2}in\pi} K_n'(ix) = H_n^{(2)}(x).$$

$K_n'(x)$ is the analytic continuation of $K_n(x)$, defined by

$$K_n'(z\, e^{m\pi i}) = e^{-mn\pi i} K_n(z), \; z \text{ real, positive.}$$

We now make the following substitutions:

$$\Delta p(x^*) = (2/\beta) \exp\left[ikM^2 x^*/\beta^2\right] \{f_1(x^*) - 2ik' f_2(x^*)\},$$

$$F(x^*) = f_1(x^*) + 4\nu'^2 f_3(x^*),$$

where $\quad f_1(x^*) = 2df_2/dx^* = 4d^2 f_3/dx^{*2}, \quad k' = k/\beta^2 \quad$ and $\quad \nu' = kM/\beta^2,$

and then integrate (22) appropriately by parts to obtain

$$\int_{-1}^{1} i\nu' F(x)\, K_1'[i\nu'(x-x_0)]\, dx = -\pi \exp\left[-ik'x_0\right] - 4\nu'^2 f_3(1) \times K_0[i\nu'(1-x_0)] + 2ik' f_2(1) \Big\{ \nu'^2/k'^2 K_0\left[i\nu'(1-x_0)\right] + (1-\nu'^2/k'^2) \int_{-1}^{\infty} i\nu' \exp\left[-ik'(1+x)\right] K_1[i\nu'(2+x-x_0)]\, dx \Big\}. \tag{24}$$

Equation (24) is the same as that governing the incompressible singularity distribution for an oblique sinusoidal gust, $w = \exp[i(\omega' t - \lambda' x - \mu' y)]$ where $\lambda' = 2k'/c$, $\mu' = 2i\nu'/c$ and $\omega' = \lambda' U$, and the load distributions are related as in (21). The difference between the two cases lies in the fact that the kernel function of (24) is actually a Hankel function while that for the incompressible oblique case is a modified Bessel function. However, since the fundamental singularity of both is the same ($1/x$ as $x \to 0$) the same method of solution can be applied to both cases.

Because of this fundamental similarity between the two cases it is reasonable to expect that Reissner's (1951) analytical solution for compressible two-dimensional problems might be applicable to the incompressible oblique problem. Although in the latter case the potential equation is no longer a wave equation, transformation to elliptic co-ordinates does still yield two Mathieu equations of the form

$$F_{\xi\xi} - (q + \chi^2 \cosh^2 \xi) F = 0$$

and

$$G_{\zeta\zeta} + (q + \chi^2 \cos^2 \zeta) G = 0.$$

In these equations q is the separation constant and the parameter $(-\chi^2)$ is negative as opposed to the positive parameter which occurs when the equations derive from a wave equation. Solutions can be found to these equations satisfying the appropriate boundary conditions. The analysis follows Reissner's except that the solutions are in terms of Mathieu functions with negative parameters and the upwash is of the convected rather than flutter type. This possible extension of Reissner's solution is only mentioned here briefly since the integral equation method appears to be the faster for the actual computation of load distributions.

4. Calculated values of lift coefficient

Using the method of solution of the integral equation (24) developed for the incompressible oblique gust, some values of the induced lift coefficient per unit upwash, C_L, have been computed for the compressible two-dimensional gust at Mach numbers of 0, 0·2, ..., 0·8. The results are shown in figure 4. The zero Mach number values agree with the solution given by Sears (1941). Those for 0·2 and 0·4 illustrate the effect of compressibility even at small values of Mach number as

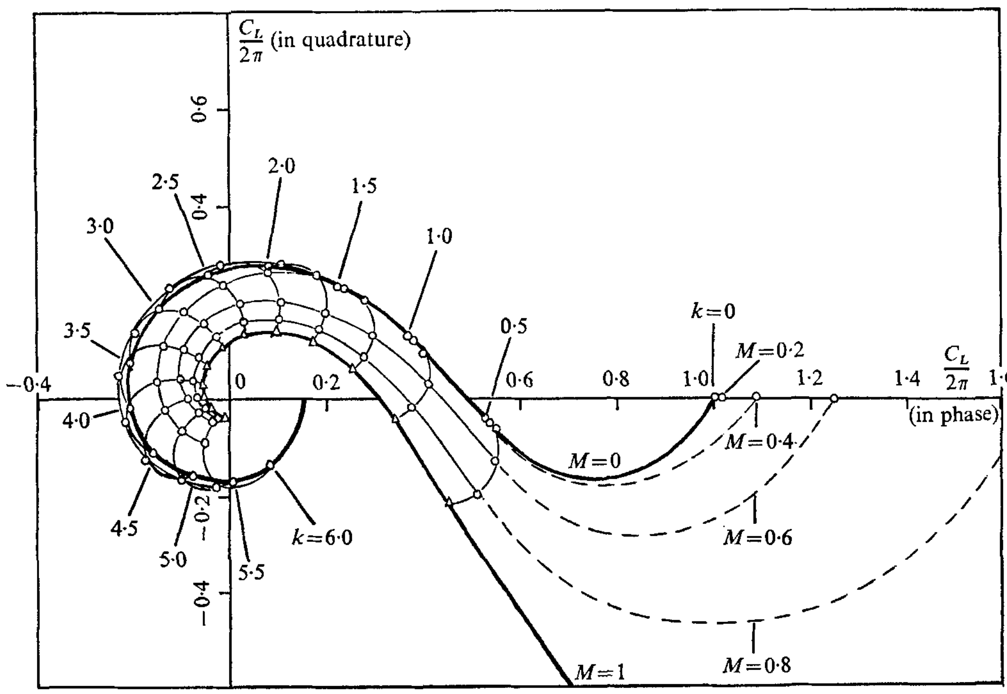

FIGURE 4. The effect of compressibility on Sears's lift function. —○—, computed values; —△—, some values from equation (25).

the parameter Mk increases. For values of this parameter not too large and for $M^2 \ll 1$, compressibility appears more to affect the phase of the lift function in the form of a lag than the amplitude, except in the quasi-static region. Also for any given finite, non-zero value of k the maximum amplitude of the induced lift coefficient occurs at a Mach number different from 0 and 1. As $M \to 1$ the computed values of lift coefficient approach the sonic value

$$C_L = \{2(1-i)/k\}.\exp(ik)\{\mathrm{C}([2k/\pi]^{\frac{1}{2}}) - i\mathrm{S}([2k/\pi]^{\frac{1}{2}})\}, \tag{25}$$

where C and S are Fresnel cosine and sine integrals. This linearized solution is valid for $M = 1$ (Landahl 1961) provided

$$k \gg |w/U|^{\frac{2}{3}}.$$

By calculating in a similar way the appropriate load distributions, compressible two-dimensional or incompressible oblique, and employing the similarity rules stated above, it is possible to obtain the response of a wing of infinite span to an arbitrarily oblique sinusoidal gust at any subsonic Mach number under conditions for which linearization is realistic. From such results the induced loading can be evaluated for an arbitrary convected gust by Fourier superposition and more importantly the spectrum of the loading induced by turbulence. The same results can also be used in the manner detailed in another paper (Graham 1970*b*) to compute the loading induced on a rectangular planform wing of arbitrary aspect ratio under similar circumstances.

5. Conclusions

Similarity rules have been given for the loading induced on a thin two-dimensional aerofoil passing subsonically through an oblique sinusoidal gust. By means of these rules all those cases, for which the gust parameter $\theta\ (= kM/\nu\beta)$ takes values less than one, can be related to an incompressible oblique problem whose solution is known. Similarly all the other cases, for which $\theta > 1$, can be related to a compressible two-dimensional problem. An efficient method of solving this latter problem has been indicated and some values of lift coefficient calculated to illustrate the effect of compressibility on the lift coefficient induced by a two-dimensional sinusoidal gust. The similarity rules given are also applicable to linear cascades and two-dimensional swept wings.

REFERENCES

CICALA, P. 1951 *NACA Tech. Memo.* 1277.

FETTIS, H. E. 1952 *USAF Wright Air Development Center, Tech. Report* no. 52-56.

FILOTAS, L. T. 1969 *Toronto University, UTIAS Report* no. 139.

GRAHAM, J. M. R. 1970*a* *Aero. Quart.* **21**, 182.

GRAHAM, J. M. R. 1970*b* Submitted to *Aero. Quart.*

HUNT, J. C. R. 1969 Internal report to *Central Electricity Research Lab. Leatherhead.*

LANDAHL, M. T. 1961 *Unsteady transonic flow.* Pergamon.

MILES, J. W. 1959 *Potential Theory of Unsteady Supersonic Flow.* Cambridge University Press.

POSSIO, C. 1938 *L'Aerotecnica*, t XVIII, fasc. 4. (Also as *British Ministry of Aircraft Production, R.T.P. Trans.* 987).

REISSNER, E. 1951 *NACA Tech. Note* 2363.

SEARS, W. R. 1941 *J. Aero. Sci.* **8**, 104.

WATKINS, C. E., RUNYAN, H. L. & WOOLSTON, D. S. 1955 *NACA Report* no. 1234.